

A numerical study of the r-mode instability of rapidly rotating nascent neutron stars

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ABSTRACT

First results of numerical analysis of classical r-modes of *rapidly* rotating compressible stellar models are reported. The full set of linear perturbation equations of rotating stars in Newtonian gravity are numerically solved without the slow rotation approximation. A critical curve of gravitational wave emission induced instability which restricts the rotational frequencies of hot young neutron stars is obtained. Taking the standard cooling mechanisms of neutron stars into account, we also show the ‘evolutionary curves’ along which neutron stars are supposed to evolve as cooling and spinning-down proceed. Rotational frequencies of $1.4M_{\odot}$ stars suffering from this instability decrease to around 100Hz when the standard cooling mechanism of neutron stars is employed. This result confirms the results of other authors who adopted the slow rotation approximation.

Key words: stars: neutron – stars: rotation – stars: oscillations

1 INTRODUCTION

Recently the r-mode instability induced by gravitational wave emission was discovered (Andersson 1998; Friedman & Morsink 1998). It is an axial mode variant of the secular instability of stellar oscillations excited by coupling with a radiation field (Chandrasekhar 1970; Friedman & Schutz 1978).

The r-mode oscillations are the extension of Rossby-Haurwitz wave of a rotating fluid known in geophysical literatures to global stellar oscillations (Papaloizou & Pringle 1978). As they are dominated by the tangential velocity perturbations, their coupling with gravitational radiation was once considered to be weaker than those of the polar modes (e.g. f-modes). Thus it was so striking and stimulating to the astrophysical community that many people began to study whether or not the instability may be strong enough to set a severe limit on the rotational period of neutron stars. In particular, several authors (Lindblom et al. 1998; Owen et al. 1998; Andersson et al. 1999a; Lockitch & Friedman 1999; Lindblom et al. 1999; Yoshida & Lee 2000) have investigated its effect on the evolution of the stellar rotational frequencies (see Friedman & Lockitch (2000) for a recent review). Those works indeed have revealed that the instability is strong

enough to limit severely the rotational frequencies of hot young neutron stars which are born with the initial temperature of $\sim 10^{11}$ K. Even if the neutron stars are born with the nearly Keplerian (mass-shedding limit) rotation frequency, they settle down to states with only a small fraction of their original angular velocity, i.e. with less than 10 % of the Keplerian rotational frequency, when they cool down to the neutron superfluid transition temperature, $T_c \sim 10^9$ K. As for the relatively cold and old neutron stars as those in X-ray binaries or radio pulsars, this instability may also set a severe limit on their rotational frequencies (Bildsten 1998; Andersson et al. 1999b; Levin 1999).

Thus far, these results have been obtained by extrapolating the results of the slow rotation approximation (Papaloizou & Pringle 1978; Provost et al. 1981; Saio 1982) to rotating models with the Keplerian frequency, except that Lindblom and Ipser (1999) obtained analytically the eigenfrequencies and the eigenfunctions of the ‘classical’ as well as the ‘generalized’ r-modes for the Maclaurin spheroids. Moreover, Bildsten & Ushomirsky (2000) recently studied a damping effect of a viscous boundary layer formed at the stellar crust–core interface. According to their work, the viscous boundary layer stabilizes the system significantly and the upper limit of the rotational frequency of neutron stars may become around ~ 500 Hz. If this is the case, the mode analysis without the slow rotation approximation should be needed.

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Thus in order to see whether the evolutionary picture mentioned above is quantitatively correct and to investigate the evolution of stellar spins, it is indispensable to study the r-modes by taking rapid rotation into account.

In this *Letter*, we will show the first numerical results of the ‘classical’ r-modes for rapidly rotating stars. Here classical r-modes are oscillations with $l = m$, where l and m are the indices of spherical harmonics. Our numerical scheme is the improved version of the one used in the polar mode analysis (Yoshida & Eriguchi 1995) and will be published elsewhere together with more detailed results for rapidly rotating compressible stars (Karino et al. 2000).

2 EVALUATION OF DAMPING TIMES

Once eigenfrequencies and eigenfunctions are obtained, timescales of the change of the system due to gravitational radiation and viscosity can be estimated by the standard procedure (Lindblom et al. 1998).

The canonical energy E_c of the perturbation is defined as:

$$E_c \equiv \frac{1}{2} \int d^3x \cdot [\rho \delta \vec{v} \cdot \delta \vec{v}^* + (\delta p / \rho - \delta \phi)^* \delta \rho], \quad (1)$$

where $\delta \rho$, δp , $\delta \vec{v}$, $\delta \phi$ are the Eulerian perturbations of the density, pressure, velocity and gravitational potential, respectively. The superscript, *, means the complex conjugation of the corresponding quantity.

When we assume that the perturbed quantities behave as $\sim e^{-i(\omega t - m\varphi)}$, the energy dissipation rate due to gravitational radiation is expressed by the multipole radiation formula as follows:

$$\dot{E}_{gr} = (\omega - m\Omega) \sum_{l \geq m} N_l \omega^{2l+1} (|\delta D_{lm}|^2 + |\delta J_{lm}|^2), \quad (2)$$

where δD_{lm} and δJ_{lm} are the mass and the mass current multipoles, respectively, defined as:

$$\delta D_{lm} = \int d^3x \cdot \delta \rho r^l Y_{lm}^*, \quad (3)$$

$$\delta J_{lm} = \frac{2}{c} \sqrt{\frac{l}{l+1}} \int d^3x \cdot r^l (\rho \delta v + \delta \rho v) \cdot \vec{Y}_{lm}^{B*}, \quad (4)$$

where c is the speed of light. Here the magnetic type vector spherical harmonics \vec{Y}_{lm}^{B*} is defined as:

$$\vec{Y}_{lm}^B = [l(l+1)]^{-\frac{1}{2}} \vec{r} \times \nabla Y_{lm}. \quad (5)$$

The coupling constant N_l is a function of l as follows:

$$N_l = \frac{4\pi G}{c^{2l+1}} \frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}. \quad (6)$$

As the rapidly rotating stars are highly deformed from spherical configuration, the summation (2) in the index l should include theoretically the infinite numbers of multipole terms. We take the lowest five ones into account here for simplicity. It should be noted that the slow rotation approximation includes only the lowest order term. On the other hand, the energy dissipation rate due to the fluid viscosity is written as:

$$\dot{E}_s = - \int d^3x \cdot 2\eta \delta \sigma^{ab} \delta \sigma_{ab}^*, \quad (7)$$

for the shear viscosity with a coefficient η , and

$$\dot{E}_b = - \int d^3x \cdot \zeta |\delta \Theta|^2, \quad (8)$$

for the bulk viscosity with a coefficient ζ . Here the volume expansion rate $\delta \Theta$ and the shear tensor $\delta \sigma_{ab}$ are defined as:

$$\delta \Theta \equiv \nabla_c \delta v^c, \quad (9)$$

and

$$\delta \sigma_{ab} \equiv \nabla_{(a} \delta v_{b)} - \frac{\delta \Theta}{3} \delta_{ab}, \quad (10)$$

where a round bracket means symmetrization among indices. These energy dissipations and their rates are numerically evaluated for the rotating stellar models.

Then the imaginary part of the eigenfrequency of the r-mode, τ_r^{-1} , is written as follows:

$$\tau_r^{-1} = \frac{\dot{E}_{gr} + \dot{E}_s + \dot{E}_b}{2E_c} \equiv \tau_{gr}^{-1} + \tau_s^{-1} + \tau_b^{-1}. \quad (11)$$

Positive values of τ_r^{-1} correspond to the states where the r-mode instability dominates the stabilizing viscous effect. Consequently critical states of the instability are defined by $\tau_r^{-1} = 0$.

3 COMPARISON WITH THE SLOW-ROTATION APPROXIMATION

We have studied the adiabatic perturbations of the equilibrium sequence of ‘canonical neutron stars’, i.e. polytropes with $p = K\rho^2$ whose mass and radius in the spherical limit become $1.4M_\odot$ and 12.5km, respectively. Here p , ρ and K are the pressure, the density and a polytropic constant, respectively. This sequence is the same as that analyzed by many authors in the slow rotation approximation. In Fig 1, the eigenfrequency ω normalized by the angular velocity Ω is plotted against the normalized angular velocity $\Omega/\sqrt{4\pi G\rho_c}$ where ρ_c and G are the central density and the gravitational constant, respectively.[†] To compare our results with those by the slow rotation approximation, we also plot the solutions by Yoshida and Lee (Yoshida & Lee 2000) where the third order term in the Ω expansion is taken into account. The result of the slowest rotation in our results ($\Omega/\sqrt{4\pi G\rho_c} = 0.0154$) agrees with theirs within the relative error of 0.4%. The grid point numbers used in this computation are 32 for the radial coordinate r from the center to the surface of the star, and 11 for the angular coordinate θ in the quarter of the meridional cross section of the star.

[†] We do not display the results of our computation in the vicinity of the mass-shedding limit, because the eigenfunctions are rather ill-behaved and the estimate of the instability time-scale is difficult there. Our results (Fig. 1 and Fig. 2) are terminated at the model with 98.7% of the mass-shedding limit frequency (850Hz).

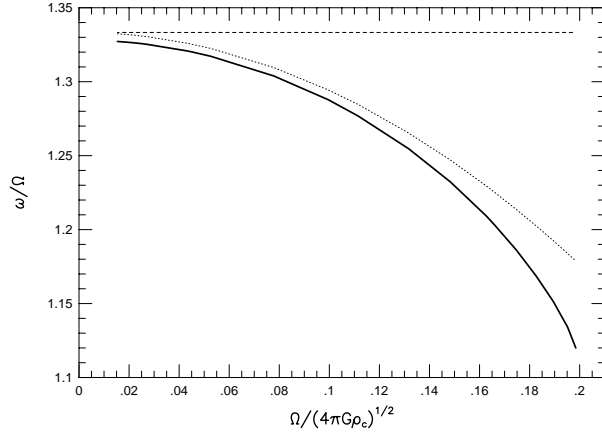


Figure 1. The normalized eigenfrequency ω/Ω is plotted against the normalized angular velocity $\Omega/(4\pi G\rho_c)^{1/2}$. Also plotted are the solutions by the slow rotation approximation. The dashed line is the solution of the first order in Ω , whereas the dotted line is the one in which the third order correction is also taken into account. The mass-shedding limit of this sequence is the point where $\Omega/(4\pi G\rho_c)^{1/2} = 0.205$

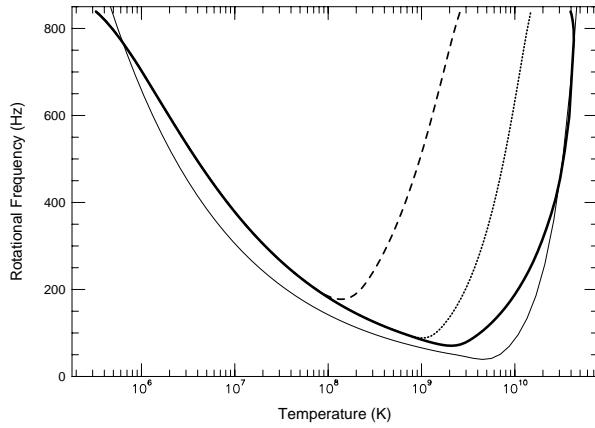


Figure 2. Critical rotational frequency of a $1.4M_\odot$ neutron star with the polytropic equation of state $p = K\rho^2$ is plotted against the temperature by the thick solid curve for the $l = m = 2$ mode. For comparison the critical curve computed using the slow rotation approximation is drawn as well (thin solid line). Also shown are the curves on which the cooling time-scale of the star is equal to the r-mode instability time-scale (dotted line for the modified URCA process; dashed line for the ‘quasi-particle’ β -decay in pion condensation).

4 CRITICAL CURVE AND ‘EVOLUTIONARY TRACK’ OF A HOT AND YOUNG NEUTRON STAR

As for the source of shear and bulk viscosity, we adopt the same microphysical process as that used by Ipser and Lindblom (Ipser & Lindblom 1991); that is, if the stellar

temperature T satisfies the relation $T > T_c$, the dominant processes are neutron-neutron (nn) collision and electron-electron (ee) collision. They result in the shear viscosity coefficient with the following density and temperature dependence:

$$\eta_{nn} \propto \rho^{9/4} T^{-2}, \quad (12)$$

and

$$\eta_{ee} \propto \rho^2 T^{-2}. \quad (13)$$

When $T \leq T_c$, only the electron collision is included. The bulk viscosity arises from lag of the β -reaction to the oscillation. The viscosity coefficient has the following dependence:

$$\zeta \propto \rho^2 (\omega - m\Omega)^{-2} T^6. \quad (14)$$

For each mode, the inverse of the time-scale τ_r^{-1} depends on the stellar temperature T as well as on the stellar rotational frequency f . In Fig 2, we plot the critical points of stability on the $T - f$ plane for the $l = m = 2$ mode. Above this critical curve, gravitational wave emission instability dominates the stabilization effect due to viscosity. Along the fixed temperature line, i.e. vertical line, the instability time-scale decreases approximately as $\sim f^{2m+2}$. As seen from this figure, the smallest value of the rotational frequency of the critical curve is roughly 8 % of the frequency at the mass-shedding limit.

Also shown is the critical curve using the slow rotation approximation. The expression of the timescales is adopted from Lindblom et al. (1999). It is noted that the shear viscosity term (Eq.(10)) is evaluated in our analysis by numerical differentiation of the corresponding eigenfunction. This leads to the relative error of $\sim 10\%$ in the shear viscous timescale τ_s for the present mesh size (32 and 11 in the r and θ direction).

Together with it plotted are the curves on which instability time-scale is equal to the cooling time-scale of the star. If we assume the ‘standard’ modified URCA process (Shapiro & Teukolsky 1983) to be dominant in the initial stage of neutron star cooling,[‡] the effective cooling time-scale τ_{cool} at the temperature T (K) is defined by:

$$\tau_{\text{cool}} = \left[\frac{d \ln T_9}{dt} \right]^{-1} = \frac{6t_c}{T_9^6}, \quad (15)$$

where $T_9 = T/10^9$ and t_c is a constant characterizing the cooling time which is typically ~ 1 yr for this process.

Apart from the early phase, hot young neutron stars may evolve along this evolutionary curve $\tau_r = \tau_{\text{cool}}$. Suppose that a neutron star is born at the temperature $\sim 10^{11}$ K and with the rotational frequency above the bottom of the critical curve. At the beginning it is stable against the r-mode instability and evolves along the horizontal line on the $T - f$ plane. In a few second, it enters the unstable region against the r-mode and the initial (stochastic) perturbation begins to grow. As long as $\tau_r > \tau_{\text{cool}}$, it evolves almost horizontally. Once it reaches

[‡] Though nucleon bremsstrahlung is also one of the standard cooling channels, its contribution is order of magnitude smaller than that of the modified URCA process. See Friman & Maxwell (1979).

the ‘evolution curve’, the star begins to go down along this curve, since any hypothetical displacement of the star from this curve will be amended by the growth of instability or the cooling. If the star is located at the right side of this curve, the star will evolve nearly horizontally leftwards due to rapid cooling. If the star is located at the left side of this curve, the star will evolve nearly vertically downward due to rapid loss of gravitational waves.

When the star goes down along the evolution curve and approaches the critical curve asymptotically, the instability ceases to work effectively and the star will not be spun down by the instability any more. The location of this evolution curve in the $T - f$ plane is insensitive to the value of constant t_c . Consequently every neutron star seems to settle down to an almost universal terminal rotational frequency, i.e. $\sim 100\text{Hz}$ at the end of the spin evolution. If $t_c = 1\text{y}$, this frequency is 90Hz and is reached when the temperature of the star equals to $1.1 \times 10^9\text{K}$.

On the other hand, if some exotic rapid cooling mechanisms dominate, the picture may be changed drastically. In Fig 2, an evolution curve defined by $\tau_r = \tau_{\text{cool}}$ for the model with pion condensation is also plotted (dashed curve). The pion condensation enhances the cooling because a new process analogous to the ordinary URCA process (Shapiro & Teukolsky 1983) begins to operate. The effective cooling time-scale for this process is written as follows:

$$\tau_{\text{cool}} = \frac{4t'_c}{T_9^4}, \quad (16)$$

where $t'_c \sim 200\text{s}$. The evolution curve of $\tau_r = \tau_{\text{cool}}$ shifts upward and the terminal rotational frequency also shifts to about 200Hz .

Commenting on an evolution of a star by a rapid cooling mechanism may be appropriate here. By an exotic cooling mechanism such as the pionic reaction, the core fluid of a hot and young neutron star cools rapidly down to $\sim 10^8\text{K}$, while the crust component of it remains hot ($\sim 10^9\text{K}$) in the first few decades (see Lattimer et al. (1994) for instance). Then the standard assumption in the r-mode instability that the stellar temperature remains uniform is broken. Andersson et al. (1999a) argues that the warm crust work as a heat reservoir and the rapid cooling effect on the stellar evolution by the r-mode is moderated. This reduces the difference between the outcome of the rapid and the standard cooling. We further note that the viscous heating by the r-mode shear motion (Levin 1999), which is omitted here and in the other works dealing with hot and young neutron stars, may delay the cooling and have the same effect as the crust. This heating prolongs the lapse during which the r-mode instability is effective (Yoshida, unpublished).

This means that in order to know the significant effect of the r-mode instability on spin evolution of newly born hot neutron stars, we need to know more about the nature of dense matter at high and intermediate temperatures.

5 SUMMARY

We show the first results of the numerical mode analysis of the classical r-mode for the *rapidly rotating and compressible* stellar models. The eigenfrequencies and their eigenfunction of the rapidly rotating stellar models are numerically computed nearly to the Kepler limit. The time scale of the r-mode instability is evaluated by the eigenfrequency and the eigenfunction.

The behaviour of the eigenfrequency is confirmed to be well approximated by the third order expansion in the angular frequency Ω . The difference between the critical curves drawn by the full computation and by the slow rotation approximation is shown to be rather small. This certifies the use of the slow rotation approximation in the third order to investigate the r-mode instability.

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